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On a class of (a, k)-regularized C-resolvent families

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Abstract. In this paper, we investigate the basic structural properties of (analytic) q-exponentially equicontinuous (a, k)-regularized C-resolvent families in sequentially complete locally convex spaces. We provide some applications in fluid dynamics, linear thermoviscoelasticity, and illustrate obtained theoretical results by several other examples.

Mathematics Subject Classification: 47D03, 47D06, 47D99.

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1. Introduction and preliminaries

In the last decades, a considerable focus on Volterra integro-differential equations and fractional calculus has been stimulated by the variety of their applications in engineering, physics, chemistry and other sciences ([3], [11], [17], [19]-[20], [23]). The class of q-exponentially equicontinuous $(C_0, 1)$-semigroups was introduced by V. A. Babalola in [2] (cf. also [4], [9]-[10] and [24]), and the purpose of our study is to examine the possibility of extension of the results obtained in this paper to abstract Volterra equations and abstract time-fractional equations. In such a way, we continue our previous work contained in [12]-[16].

The paper is organized as follows. In Theorem 2.1-Theorem 2.2, we generalize the subordination principle for abstract time-fractional equations, and the abstract Weierstrass formula (cf. [12, Theorem 3.9, Theorem 3.21]).

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In the third section of the paper, we analyze some generation results for q-exponentially equicontinuous (a,k)-regularized C-resolvent families in complete locally convex spaces (notice that the completeness of underlying locally convex space E is not used in the second section as well as in the formulation of Theorem 3.1(i)). Although the restriction $C = I$ seems inevitable here, it is not clear whether the condition $k(0) = 0$, used only in the proof of non-degeneracy of the (a,k)-regularized resolvent family $(\overline{R_p}(t))_{t\geq 0}$ appearing in the formulation of Theorem 3.1(i), is superfluous (for further information in this direction, we refer the reader to [19, Proposition 2.5] and [12, Proposition 2.4(ii)]). Theorem 3.1 is the main result of the paper and has several obvious consequences of which we will emphasize only the most significant perturbation type theorems (cf. Theorem 3.2 and Example 3.1).

Throughout the paper, we assume that E is a Hausdorff sequentially complete locally convex space, SCLCS for short, and that the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E. By $L(E)$ we denote the space which consists of all continuous linear mappings from E into E. The domain and the resolvent set of a closed linear operator A acting on E are denoted by $D(A)$ and $\rho(A)$, respectively. We use the notation $D_\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$. Suppose F is a linear subspace of E. Then the part of A in F, denoted by $A_{|F}$, is the linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Let $C \in L(E)$ be injective. Then the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A$ is injective and $(\lambda - A)^{-1}C \in L(E)\}$.

For every $p \in \circledast$, we define the factor space $E_p \equiv E/p^{-1}(0)$. The norm of a class $x + p^{-1}(0)$ is defined by $||x + p^{-1}(0)||_{E_p} := p(x)$ $(x \in E)$. Then the canonical mapping $\Psi_p : E \to E_p$ is continuous; the completion of E_p under the norm $||\cdot||_{E_p}$ is denoted by $\overline{E_p}$. Since no confusion seems likely, we also denote the norms on E_p and $L(E_p)$ ($\overline{E_p}$ and $L(\overline{E_p})$) by $||\cdot||$; $L_\circledast(E)$ denotes the subspace of $L(E)$ which consists of those bounded linear operators T on E such that, for every $p \in \circledast$, there exists $c_p > 0$ satisfying $p(Tx) \leq c_p p(x)$, $x \in E$. The infimum of such numbers c_p, denoted by $P_p(T)$, satisfies $P_p(T) = \sup_{x\in E, p(x)\leq 1} p(Tx)$ $(p \in \circledast)$. It is clear that $P_p(T_1T_2) \leq P_p(T_1)P_p(T_2)$, $p \in \circledast$, $T_1,\ T_2 \in L_\circledast(E)$ and that $P_p(\cdot)$ is a seminorm on $L_\circledast(E)$. If $T \in L_\circledast(E)$ and $p \in \circledast$, then the operator $T_p : E_p \to E_p$, defined by $T_p(\Psi_p(x)) := \Psi_p(Tx)$, $x \in E$, belongs to $L(E_p)$. Moreover, the operator T_p can be uniquely extended to a bounded linear operator $\overline{T_p}$ on $\overline{E_p}$ and the following holds: $||T_p|| =$

$||\overline{T_p}|| = P_p(T)$. Define $V_p := \{x \in E : p(x) \leq 1\}$ ($p \in \circledast$) and order $\circledast$ by: $p \gg q$ iff $V_p \subseteq V_q$ ($p,\ q \in \circledast$). The function $\pi_{qp} : E_p \to E_q$, defined by $\pi_{qp}(\Psi_p(x)) := \Psi_q(x)$, $x \in E$, is a continuous homomorphism of E_p onto E_q, and extends therefore, to a continuous linear homomorphism π_{qp} of $\overline{E_p}$ onto $\overline{E_q}$. The reader may consult [2] for the basic facts about projective limits of Banach spaces (closed linear operators acting on Banach spaces) and their projective limits.

Given $s \in \mathbb{R}$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $0^\alpha := 0$ and $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ ($\alpha > 0$, $t > 0$). If $\delta \in (0, \pi]$, then we define $\Sigma_\delta := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg \lambda| < \delta\}$. We refer the reader to [22, pp. 99–102] for the basic material concerning integration in SCLCSs, and to [12] for the definition and elementary properties of analytic functions with values in SCLCSs.

We need the following definition from [11]-[12].

Definition 1.1.

(i) Let $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $k \neq 0$ and let $a \in L^1_{loc}([0, \tau))$, $a \neq 0$. A strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) (a, k)-regularized C-resolvent family having A as a subgenerator iff the following holds:

(i.1) $R(t)A \subseteq AR(t),\ t \in [0, \tau)$, $R(0) = k(0)C$ and $CA \subseteq AC$,

(i.2) $R(t)C = CR(t),\ t \in [0, \tau)$ and

(i.3) $R(t)x = k(t)Cx + \int_0^t a(t - s)AR(s)x\,ds,\ t \in [0, \tau),\ x \in D(A)$;

$(R(t))_{t\in[0,\tau)}$ is said to be non-degenerate if the condition $R(t)x = 0,\ t \in [0, \tau)$ implies $x = 0$, and $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous if, for every $t \in (0, \tau)$, the family $\{R(s) : s \in [0, t]\}$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous.

(ii) Let $\beta \in (0, \pi]$ and let $(R(t))_{t\geq 0}$ be an (a, k)-regularized C-resolvent family. Then it is said that $(R(t))_{t\geq 0}$ is an analytic (a, k)-regularized C-resolvent family of angle β, if there exists a function $\mathbf{R} : \Sigma_\beta \to L(E)$ satisfying that, for every $x \in E$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_\beta$ is analytic as well as that:

(ii.1) $\mathbf{R}(t) = R(t),\ t > 0$ and

(ii.2) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{R}(z)x = k(0)Cx$ for all $\gamma \in (0,\beta)$ and $x \in E$.

It is said that $(R(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family, resp. equicontinuous analytic (a,k)-regularized C-resolvent family of angle β, if for every $\gamma \in (0,\beta)$, there exists $\omega_\gamma \geq 0$, resp. $\omega_\gamma = 0$, such that the set $\{e^{-\omega_\gamma |z|}\mathbf{R}(z) : z \in \Sigma_\gamma\}$ is equicontinuous. Since there is no risk for confusion, we will identify in the sequel $R(\cdot)$ and $\mathbf{R}(\cdot)$.

In the case $k(t) = g_{\alpha+1}(t)$, where $\alpha > 0$, it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; in such a way, we unify the notions of (local) α-times integrated C-semigroups ($a(t) \equiv 1$) and cosine functions ($a(t) \equiv t$) in locally convex spaces ([6], [18], [27]). Furthermore, in the case $k(t) = \int_0^t K(s)ds,\ t \in [0,\tau)$, where $K \in L^1_{loc}([0,\tau))$ and $K \neq 0$, we obtain the unification concept for (local) K-convoluted C-semigroups and cosine functions ([13]). If $C = I$, then $(R(t))_{t\in[0,\tau)}$ is also said to be an (a,k)-regularized resolvent family with a subgenerator A ([8], [11]-[12], [19]). From now on, we always assume that $a \neq 0$ in $L^1_{loc}([0,\tau))$ and that $K,\ k,\ k_1,\ k_2, \cdots$ are scalar-valued kernels; all considered (a,k)-regularized C-resolvent families will be non-degenerate.

Let $a(t)$ be a kernel. Then one can define the integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ by setting

(1)
$$\hat{A} := \Bigl\{(x,y) \in E \times E : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)yds,\ t \in [0,\tau)\Bigr\}.$$

The integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ is a linear operator in E which extends any subgenerator of $(R(t))_{t\in[0,\tau)}$ and satisfies $C^{-1}\hat{A}C = \hat{A}$. The local equicontinuity of $(R(t))_{t\in[0,\tau)}$ guarantees that $\hat{A}$ is a closed linear operator in E; if, additionally,

$$A\int_0^t a(t-s)R(s)xds = R(t)x - k(t)Cx,\ t \in [0,\tau),\ x \in E, \tag{2}$$

then $R(t)R(s) = R(s)R(t)$, $t,\ s \in [0,\tau)$ (cf. [15]) and $\hat{A}$ is a subgenerator of $(R(t))_{t\in[0,\tau)}$. For more details on subgenerators of (a,k)-regularized C-resolvent families, the reader may consult [12]-[13].

The following condition will be used frequently:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}k(t)dt := \int_0^\infty e^{-\lambda t}k(t)dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$.

Put $abs(k)$:=inf$\{\Re\lambda : \tilde{k}(\lambda)$ exists$\}$ and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

Let $\alpha > 0$, let $\beta \in \mathbb{R}$ and let the Mittag-Leffler function $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n+\beta)$, $z \in \mathbb{C}$. In this place, we assume that $1/\Gamma(\alpha n+\beta) = 0$ if $\alpha n+\beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. The Wright function $\Phi_\gamma(t)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geq 0$. As is well-known, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that:

$$E_\alpha(t) \leq c_\alpha \exp\bigl(t^{1/\alpha}\bigr),\ t \geq 0. \tag{3}$$

Henceforth $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative of order α ([3]).

The asymptotic expansion of the entire function $E_{\alpha,\beta}(z)$ is given in the following lemma (cf. [26, Theorem 1.1]):

Lemma 1.1. *Let $0 < \sigma < \frac{1}{2}\pi$. Then, for every $z \in \mathbb{C}\setminus\{0\}$ and $m \in \mathbb{N}\setminus\{1\}$:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_s Z_s^{1-\beta}e^{Z_s} - \sum_{j=1}^{m-1}\frac{z^{-j}}{\Gamma(\beta-\alpha j)} + O(|z|^{-m}),\ |z| \to \infty,$$

where Z_s is defined by $Z_s := z^{1/\alpha}e^{2\pi i s/\alpha}$ and the first summation is taken over all those integers s satisfying $|\arg z + 2\pi s| < \alpha(\frac{\pi}{2}+\sigma)$.

For further information concerning Mittag-Leffler and Wright functions, we refer the reader to [3, Section 1.3].

2. Q-exponentially equicontinuous (a,k)-regularized C-resolvent families

We introduce (analytic) q-exponentially equicontinuous (a,k)-regularized C-resolvent families as follows.

Definition 2.1.

(i) Let $k \in C([0,\infty))$ and let $a \in L^1_{loc}([0,\infty))$. Suppose that $(R(t))_{t\geq 0}$ is a global (a,k)-regularized C-resolvent family having A as a subgenerator. Then it is said that $(R(t))_{t\geq 0}$ is a quasi-exponentially equicontinuous (q-exponentially equicontinuous, for short) (a,k)-regularized C-resolvent family having A as a subgenerator iff, for every $p \in \circledast$, there exist $M_p \geq 1$, $\omega_p \geq 0$ and $q_p \in \circledast$ such that:

$$p(R(t)x) \leq M_p e^{\omega_p t} q_p(x),\ t \geq 0,\ x \in E. \tag{4}$$

If, for every $p \in \circledast$, one can take $\omega_p = 0$, then $(R(t))_{t\geq 0}$ is said to be an equicontinuous (a,k)-regularized C-resolvent family.

(ii) Let $\beta \in (0,\pi]$ and let A be a subgenerator of an analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle β. Then it is said that $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family of angle β, if for every $p \in \circledast$ and $\epsilon \in (0,\beta)$, there exist $M_{p,\epsilon} \geq 1$, $\omega_{p,\epsilon} \geq 0$ and $q_{p,\epsilon} \in \circledast$ such that:

$$p(R(z)x) \leq M_{p,\epsilon} e^{\omega_{p,\epsilon}|z|} q_{p,\epsilon}(x),\ z \in \Sigma_{\beta-\epsilon},\ x \in E.$$

It is clear from Definition 2.1 that every q-exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ is locally equicontinuous. On the other hand, the following example from [2] shows that $(R(t))_{t\geq 0}$ need not be exponentially equicontinuous, in general: Let $a(t) = k(t) = 1$, let $C = I$ and let the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$ be topologized by the following system of seminorms $p_{m,n}(f) := ||x^m f^{(n)}(x)||_{L^2(\mathbb{R})}$ ($m,\ n \in \mathbb{N}_0,\ f \in \mathcal{S}(\mathbb{R})$); notice that the usual topology on $\mathcal{S}(\mathbb{R})$, induced by the seminorms $q_{m,n}(f) = ||x^m f^{(n)}(x)||_{L^\infty(\mathbb{R})}$ ($m,\ n \in \mathbb{N}_0,\ f \in \mathcal{S}(\mathbb{R})$), is equivalent to the topology introduced above. Set $(S(t)f)(x) := f(e^t x)$, $t \geq 0$, $x \in \mathbb{R}$, $f \in \mathcal{S}(\mathbb{R})$. Then $(S(t))_{t\geq 0}$ is a q-exponentially equicontinuous (a,k)-regularized resolvent family (i.e., q-exponentially equicontinuous $(C_0,1)$-semigroup) whose integral generator is the bounded linear operator $A \in L(\mathcal{S}(\mathbb{R}))$ given by $(Af)(x) := xf'(x)$, $x \in \mathbb{R}$, $f \in \mathcal{S}(\mathbb{R})$; $(S(t))_{t\geq 0}$ is not exponentially equicontinous, $(S(t))_{t\geq 0}$ has no Laplace transform in $\mathcal{S}(\mathbb{R})$ and $p_{mn}(S(t)f) = e^{(n-m-(1/2))t} p_{mn}(f)$ ($t \geq 0,\ m,\ n \in \mathbb{N}_0,\ f \in \mathcal{S}(\mathbb{R})$). It can be easily proved that there does not exist an injective operator $C \in L(\mathcal{S}(\mathbb{R}))$ such that A is the integral generator of an exponentially equicontinuous C-regularized semigroup in $\mathcal{S}(\mathbb{R})$.

Let A be a subgenerator of an exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying the equality (2) for all $t \geq 0$ and $x \in E$. If $a(t)$ and $k(t)$ satisfy (P1), then one can define, with the help of Laplace transform, the integral generator $\hat{A}$ of $(R(t))_{t\geq 0}$ by (1); in case that $a(t)$ is a kernel, then the definition of integral generator $\hat{A}$ of $(R(t))_{t\geq 0}$ coincides with the corresponding one introduced in the first section ([15]). Suppose now $a(t)$ and $k(t)$ satisfy (P1) as well as $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous (a,k)-regularized C-resolvent family with a subgenerator A. We will prove that $\hat{A}$, defined by (1), is single-valued. Towards this end, assume that $y \in E$ satisfies $\int_0^t a(t-s)R(s)y\,ds = 0$, $t \geq 0$. Let $p \in \circledast$. Then, for any $\lambda > \max(abs(a), \omega_p)$,

$$\int\limits_0^{\infty} e^{-\lambda t}\Psi_p\Bigl(\int_0^t a(t-s)R(s)y\,ds\Bigr)dt = \int\limits_0^{\infty} e^{-\lambda t}\int_0^t a(t-s)\Psi_p(R(s)y)\,ds\,dt = 0.$$

By the uniqueness theorem for the Laplace transform, one yields $\Psi_p(R(t)y) = 0$, $t \geq 0$, which implies by the arbitrariness of p and the non-degeneracy of $(R(t))_{t\geq 0}$ that $R(t)y = y = 0$, $t \geq 0$. Hence, $\hat{A}$ is a linear operator in E. It readily follows that $\hat{A}$ is a closed linear operator in E which extends any subgenerator of $(R(t))_{t\geq 0}$ and satisfies $C^{-1}\hat{A}C = \hat{A}$. Let A and B be subgenerators of $(R(t))_{t\geq 0}$. Then $Ax = Bx$, $x \in D(A) \cap D(B)$, and $A \subseteq B \Leftrightarrow D(A) \subseteq D(B)$. If (2) additionally holds, then $R(t)R(s) = R(s)R(t)$, $t,\ s \geq 0$, $\hat{A}$ itself is a subgenerator of $(R(t))_{t\geq 0}$ and $\hat{A} = C^{-1}AC$. Assuming that (2) holds with A replaced by B therein, we have the following:

(i) $C^{-1}AC = C^{-1}BC$ and $C(D(A)) \subseteq D(B)$.

(ii) A and B have the same eigenvalues.

(iii) $A \subseteq B \Rightarrow \rho_C(A) \subseteq \rho_C(B)$.

The proof of following proposition is standard and as such will not be given.

Proposition 2.1.

(i) *Let $(R(t))_{t\geq 0}$ be a global exponentially equicontinuous (q-exponentially equicontinuous) (a,k)-regularized C-resolvent family with a subgenerator A and let $b \in L^1_{loc}([0,\tau))$ be a kernel. If the function $t \mapsto \int_0^t |b(s)|ds$,*

$t \geq 0$ is exponentially bounded, then A is a subgenerator of a global exponentially equicontinuous (q-exponentially equicontinuous) $(a, k * b)$-regularized C-resolvent family $((b * R)(t))_{t\geq 0}$.

(ii) *Let $(E_i)_{i\in I}$ be a family of SCLCSs and let $E := \prod_{i\in I} E_i$ be its direct product. Assume that, for every $i \in I$, $(S_i(t))_{t\geq 0}$ is a q-exponentially equicontinuous (equicontinuous) (a, k)-regularized C_i-resolvent family in E_i having A_i as a subgenerator. Set $A_i := \prod_{i\in I} A_i$, $C := \prod_{i\in I} C_i$ and $S(t) := \prod_{i\in I} S_i(t)$, $t \geq 0$. Then $(S(t))_{t\geq 0}$ is a q-exponentially equicontinuous (equicontinuous) (a, k)-regularized C-resolvent family in E, having A as subgenerator.*

(iii) *Assume $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous (equicontinuous) (a, k)-regularized C-resolvent family with a subgenerator A. Set $p_n(x) := \sum_{i=0}^{n} p(A^i x)$, $x \in D_\infty(A)$, $p \in \circledast$, $n \in \mathbb{N}$, $R_\infty(t) := R(t)_{|D_\infty(A)}$, $t \geq 0$, $A_\infty := A_{|D_\infty(A)}$ and $C_\infty := C_{|D_\infty(A)}$. Then the system $(p_n)_{p\in\circledast, n\in\mathbb{N}}$ induces a Hausdorff sequentially complete locally convex topology on $D_\infty(A)$, $A_\infty \in L(D_\infty(A))$ and $(R_\infty(t))_{t\geq 0}$ is a q-exponentially equicontinuous (equicontinuous) (a, k)-regularized C_∞-resolvent family with a subgenerator A_∞. Furthermore, the following holds: If $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous (equicontinuous), analytic (a, k)-regularized C-resolvent family of angle $\beta \in (0, \pi]$ and $R(z)A \subseteq AR(z)$, $z \in \Sigma_\beta$, then $(R_\infty(t))_{t\geq 0}$ is likewise a q-exponentially equicontinuous (equicontinuous), analytic (a, k)-regularized C-resolvent family of angle β.*

Notice that it is not clear whether the general assumptions of Proposition 2.1(iii) imply that the space $D_\infty(A)$ is non-trivial. Now we would like to observe that the assertions of [13, Theorem 2.1.27(xiii)-(xiv), Theorem 2.5.1-Theorem 2.5.3, Remark 2.5.4(iii), Theorem 2.5.5-Theorem 2.5.6] and [16, Theorem 2.1, Corollary 2.2, Theorem 2.3, Corollary 2.4] can be simply reformulated for (analytic) q-exponentially equicontinuous (a, k)-regularized C-resolvent families in SCLCSs. This is not the case with the assertions of [12, Theorem 2.14-Theorem 2.15]; even on reflexive spaces, the adjoint of a q-exponentially equicontinuous $(C_0, 1)$-semigroup need not be of the same class ([2]). Notice also that Proposition 2.1(ii) can allow one to construct some artificial examples of q-exponentially equicontinuous (not exponentially equicontinuous, in general) (a,k)-regularized C-resolvent families, with $C \neq I$ or $k(0)=0$.

The following theorem is an extension of [12, Theorem 3.9].

Theorem 2.1. *Assume $k_\beta(t)$ satisfies (P1), $0<\alpha<\beta$, $\gamma=\alpha/\beta$ and A is a subgenerator of a q-exponentially equicontinuous (g_β,k_β)-regularized C-resolvent family $(S_\beta(t))_{t\geq 0}$ satisfying (4) with $R(\cdot)$ replaced by $S_\beta(\cdot)$ therein. Assume that there exist a continuous function $k_\alpha(t)$ satisfying (P1) and a number $\upsilon>0$ such that $k_\alpha(0)=k_\beta(0)$ and*

$$\widetilde{k_\alpha}(\lambda)=\lambda^{\gamma-1}\widetilde{k_\beta}\big(\lambda^\gamma\big),\ \lambda>\upsilon. \tag{5}$$

Then A is a subgenerator of a q-exponentially equicontinuous (g_α,k_α)-regularized C-resolvent family $(S_\alpha(t))_{t\geq 0}$, given by

$$S_\alpha(t)x:=\int_0^\infty t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)S_\beta(s)x\,ds,\ x\in E,\ t>0 \text{ and } S_\alpha(0):=k_\alpha(0)C.$$

Furthermore,

$$p\big(S_\alpha(t)x\big)\leq c_\gamma M_p\exp\big(\omega_p^{1/\gamma}t\big)q_p(x),\ p\in\circledast,\ t\geq 0,\ x\in E. \tag{6}$$

Let $p\in\circledast$. Then the condition

$$p\big(S_\beta(t)x\big)\leq M_p\big(1+t^{\xi_p}\big)e^{\omega_p t}q_p(x),\ t\geq 0,\ x\in E\ \big(\xi_p\geq 0\big), \tag{7}$$

resp.,

$$p\big(S_\beta(t)x\big)\leq M_p t^{\xi_p}e^{\omega_p t}q_p(x),\ t\geq 0,\ x\in E, \tag{8}$$

implies that there exists $M_p'\geq 1$ such that

$$p\big(S_\alpha(t)x\big)\leq M_p'\big(1+t^{\xi_p\gamma}\big)\big(1+\omega_p t^{\xi_p(1-\gamma)}\big)\exp\big(\omega_p^{1/\gamma}t\big)q_p(x),\ t\geq 0,\ x\in E, \tag{9}$$

resp.,

$$p\big(S_\alpha(t)x\big)\leq M_p' t^{\xi_p\gamma}\big(1+\omega_p t^{\xi_p(1-\gamma)}\big)\exp\big(\omega_p^{1/\gamma}t\big)q_p(x),\ t\geq 0,\ x\in E. \tag{10}$$

We also have the following:

(i) *The mapping* $t \mapsto S_\alpha(t)$, $t > 0$ *admits an extension to* $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ *and, for every* $x \in E$, *the mapping* $z \mapsto S_\alpha(z)x$, $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ *is analytic.*

(ii) *Let* $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi))$. *If, for every* $p \in \circledast$, *one has* $\omega_p = 0$, *then* $(S_\alpha(t))_{t\geq 0}$ *is an equicontinuous analytic* (g_α, k_α)*-regularized* C*-resolvent family of angle* $\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)$.

(iii) *If* $\omega_p > 0$ *for some* $p \in \circledast$, *then* $(S_\alpha(t))_{t\geq 0}$ *is a* q*-exponentially equicontinuous, analytic* (g_α, k_α)*-regularized* C*-resolvent family of angle* $\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})$.

Proof. By definition of Wright function and (3), we have that (cf. also the proof of [3, Theorem 3.1]):

$$
\begin{aligned}
p\big(S_\alpha(t)x\big) &\leq q_p(x)\int_0^\infty t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)M_p e^{\omega_p s}\,ds \\
&= M_p q_p(x) E_\gamma\big(\omega_p t^\gamma\big) \leq M_p c_\gamma \exp\big(\omega_p^{1/q} t\big) q_p(x),\ p \in \circledast,\ x \in E,\ t \geq 0,
\end{aligned}
$$

which implies (6). By the proof of the above-mentioned theorem, we get that $(S_\alpha(t))_{t\geq 0}$ is strongly continuous. It can be easily seen that $S_\alpha(t)A \subseteq AS_\alpha(t)$ and $S_\alpha(t)C = CS_\alpha(t)$ $(t \geq 0)$. Let $x \in D(A)$ and $p \in \circledast$ be fixed. Using [3, (3.10)], the functional equation of $(S_\beta(t))_{t\geq 0}$ (cf. Definition 1.1(i.3) with $a(t) = g_\beta(t)$ and $k(t) = k_\beta(t)$), the Fubini theorem and the elementary properties of vector-valued Laplace transform, it follows that there exists a sufficiently large number $\kappa_p > \upsilon$ such that (the integrals are taken in the sense of convergence in $\overline{E_p}$):

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t}\Psi_p\big(S_\alpha(t)x\big)dt \\
&\quad = \int_0^\infty\int_0^\infty \Psi_p\big(e^{-\lambda t}t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)S_\beta(s)x\big)dsdt \\
&\quad = \int_0^\infty\int_0^\infty \Psi_p\big(e^{-\lambda t}t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)S_\beta(s)x\big)dtds \\
&\quad = \lambda^{\gamma-1}\int_0^\infty e^{-\lambda^\gamma s}\Psi_p\big(S_\beta(s)x\big)ds \qquad (11)\\
&\quad = \lambda^{\gamma-1}\int_0^\infty e^{-\lambda^\gamma s}\Psi_p\Big(k_\beta(s)Cx+\int_0^s g_\beta(s-r)S_\beta(r)Axdr\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda^{\gamma-1}\widetilde{k_\beta}\big(\lambda^\gamma\big)\Psi_p(Cx)+\lambda^{\gamma-1}\lambda^{-\beta\gamma}\int_0^\infty e^{-\lambda^\gamma s}\Psi_p\big(S_\beta(s)Ax\big)ds \\
&= \lambda^{\gamma-1}\widetilde{k_\beta}\big(\lambda^\gamma\big)\Psi_p(Cx)+\lambda^{-\alpha}\lambda^{\gamma-1}\int_0^\infty e^{-\lambda^\gamma s}\Psi_p\big(S_\beta(s)Ax\big)ds
\end{aligned}
$$

(12)

$$
\begin{aligned}
&= \int_0^\infty e^{-\lambda t}\Psi_p\big(k_\alpha(t)Cx\big)dt+\int_0^\infty e^{-\lambda t}\Psi_p\Big(\int_0^t g_\alpha(t-s)S_\alpha(s)Axds\Big)dt \\
&= \int_0^\infty e^{-\lambda t}\Psi_p\Big(k_\alpha(t)Cx+\int_0^t g_\alpha(t-s)S_\alpha(s)Axds\Big)dt,\ \lambda>\kappa_p,
\end{aligned}
$$

where (12) follows from (5) and (11). Therefore,

(13)

$$
\int_0^\infty e^{-\lambda t}\Psi_p\Big(S_\alpha(t)x-k_\alpha(t)Cx-\int_0^t g_\alpha(t-s)S_\alpha(s)Axds\Big)dt=0,\ \lambda>\kappa_p.
$$

By the uniqueness theorem for the Laplace transform and the fact that E is Hausdorff, we obtain from (13) that $S_\alpha(t)x = k_\alpha(t)Cx + \int_0^t g_\alpha(t-s)S_\alpha(s)Axds$, $t \geq 0$. Suppose now that $S_\alpha(t)x = 0$, $t \geq 0$ for some $x \in E$. Then, for every $p \in \circledast$, there exists a sufficiently large $\xi_p > 0$ such that (11) holds for any $\lambda > \xi_p$, which implies by the uniqueness theorem for the Laplace transform that $\Psi_p(S_\beta(t)x) = 0$, $t \geq 0$. Therefore, $S_\beta(t)x = 0$, $t \geq 0$ and $x = 0$, because $(S_\beta(t))_{t\geq 0}$ is non-degenerate. Hence, $(S_\alpha(t))_{t\geq 0}$ is a q-exponentially equicontinuous (g_α, k_α)-regularized C-resolvent family with a subgenerator A. Suppose now that (7), resp. (8), holds. Using the integral representation of the Wright Function [3, (1.30)], the Fubini theorem and the Laplace transform, it can be simply proved that there exists $M_p'' \geq 1$ such that:

$$\int_0^\infty e^{\omega_p s t^\gamma} \Phi_\gamma(s) s^{\xi_p} ds \leq M_p'' \Bigl(1 + \bigl(\omega_p t^\gamma\bigr)^{\frac{\xi_p(1-\gamma)}{\gamma}}\Bigr) \exp\bigl(\omega_p^{1/\gamma} t\bigr),$$

provided $\omega_p > 0$ and $t \geq \omega_p^{(-1)/\gamma}$. This immediately implies that (9), resp. (10), holds. The proofs of (i)-(iii) essentially follows from [12, Lemma 3.3, Theorem 3.4] and the proof of [3, Theorem 3.3]; here the only non-trivial part is the continuity of mapping $z \mapsto S_\alpha(z)x$ on closed sectors containing the non-negative real axis ($x \in E$). For the convenience of the reader, we will prove this assertion in the case that $\omega_p > 0$ for some $p \in \circledast$ (cf. (iii)). Put $\kappa_\gamma := \min((\frac{1}{\gamma} - 1)\frac{\pi}{2}, \frac{\pi}{2})$. Let $p \in \circledast$, $x \in E$ and $\delta \in (0, \kappa_\gamma)$ be fixed, and let $\delta' \in (\delta, \kappa_\gamma)$. By the proof of [3, Theorem 3.3], we infer that there exist $M_{p,\delta'} \geq 1$ and $\omega_{p,\delta'} > 0$ such that $p(S_\alpha(z)x) \leq M_{p,\delta'} e^{\omega_{p,\delta'} \Re z} q_p(x)$, $z \in \Sigma_{\delta'}$ and that the mapping $z \mapsto \langle x^*, S_\alpha(z)x\rangle$, $z \in \Sigma_{\kappa_\gamma}$ ($x^* \in E^*$) is analytic, which implies the analyticity of mapping $z \mapsto S_\alpha(z)x$, $z \in \Sigma_{\kappa_\gamma}$. Let $\xi_{p,\delta'} > \omega_{p,\delta'}$. Then the function $z \mapsto e^{-\xi_{p,\delta'} z}\Psi_p(S_\alpha(z)x)$, $z \in \Sigma_{\delta'}$ is analytic and bounded. Since $\lim_{t\downarrow 0+} \Psi_p(S_\alpha(t)x) = \Psi_p(k_\alpha(0)Cx)$, we obtain from [12, Theorem 3.4(ii)] that $\lim_{z\to 0, z\in\Sigma_\delta} \Psi_p(S_\alpha(z)x) = \Psi_p(k_\alpha(0)Cx)$. The above yields $\lim_{z\to 0, z\in\Sigma_\delta} p(S_\alpha(z)x - k_\alpha(0)Cx) = 0$, and since p is arbitrary, $\lim_{z\to 0, z\in\Sigma_\delta} S_\alpha(z)x = k_\alpha(0)Cx$. □

Combining the proof of Theorem 2.1 with [1, Lemma 1.6.7], we obtain the following slight generalization of the abstract Weierstrass formula [12, Theorem 3.21]:

Theorem 2.2.

(i) *Assume $k(t)$ and $a(t)$ satisfy (P1), and there exist $M > 0$ and $\omega > 0$ such that $|k(t)| \leq Me^{\omega t}$, $t \geq 0$. Assume, further, that there exist a number $\omega' \geq \omega$ and a function $a_1(t)$ satisfying (P1) and $\widetilde{a_1}(\lambda) = \tilde{a}(\sqrt{\lambda})$, $\Re\lambda > \omega'$. Let A be a subgenerator of a q-exponentially equicontinuous (a, k)-regularized C-resolvent family $(C(t))_{t\geq 0}$. Then A is a subgenerator of a q-exponentially equicontinuous, analytic (a_1, k_1)-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$, where:*

$$k_1(t) := \int\limits_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} k(s)ds, \ t > 0, \ k_1(0) := k(0), \ \textit{and}$$

$$R(t)x := \int\limits_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} C(s)xds, \ t > 0, \ x \in E, \ R(0) := k(0)C.$$

(ii) *Suppose $a > 0$, $\beta > 0$ and $k_{2\beta}(t)$ satisfies (P1). Let A be a subgenerator of a q-exponentially equicontinuous $(g_{2\beta}, k_{2\beta})$-regularized C-resolvent family $(R_{2\beta}(t))_{t\geq 0}$ and let $k_\beta(t)$ satisfy (P1), $k_\beta(0) = k_{2\beta}(0)$ and $\widetilde{k_\beta}(\lambda) = \lambda^{(-1)/2}\widetilde{k_{2\beta}}(\lambda^{1/2})$, $\Re\lambda > a$. Then A is a subgenerator of a q-exponentially equicontinuous, analytic (g_β, k_β)-regularized C-resolvent family $(R_\beta(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$, where:*

$$R_\beta(t)x := \int\limits_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} R_{2\beta}(s)xds, \ t > 0, \ x \in E, \ R_\beta(0) := k_\beta(0)C.$$

It is clear that Theorem 2.1-Theorem 2.2 can be applied to a class of differential operators with variable coefficients on $\mathcal{S}(\mathbb{R}^n)$ (cf. [2, Section 6] and [7]). For example, let $\mathcal{S}(\mathbb{R})$ be topologized as before and let the operator $A \in L(\mathcal{S}(\mathbb{R}))$ be defined by $(Af)(x) := x^2 f''(x) + xf'(x)$, $x \in \mathbb{R}$, $f \in \mathcal{S}(\mathbb{R})$. Then A is the integral generator of a q-exponentially equicontinuous cosine function $(C(t)\diamond \equiv \frac{1}{2}(\diamond(e^t\cdot) + \diamond(e^{-t}\cdot)))_{t\geq 0}$ in $\mathcal{S}(\mathbb{R})$, which implies by Theorem 2.1 that, for every $\alpha \in (0, 2)$, the operator A is the integral generator of a q-exponentially equicontinuous, analytic (g_α, g_1)-regularized resolvent family of angle $\delta_\alpha \equiv \min((\frac{2}{\alpha} - 1)\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, for every $\alpha \in (0, 2)$, the abstract Cauchy problem:

$$\mathbf{D}_t^\alpha u(t, x) = x^2 u_{xx}(t, x) + xu_x(t, x), \ t > 0, \ x \in \mathbb{R};$$

$$u(0,x)=f_0(x),\ \text{and}\ u_t(0,x)=f_1(x)\ \text{if}\ \alpha\in(1,2),$$

has a unique solution for any $f_0,\ f_1 \in \mathcal{S}(\mathbb{R})$, and the mapping $t \mapsto u(t,\cdot) \in \mathcal{S}(\mathbb{R})$, $t > 0$ is analytically extensible to the sector Σ_{δ_α} ([13]). Furthermore, Theorem 3.1 stated below and [13, Theorem 2.4.19] together imply that, for every $\alpha \in (0,1)$, A is the integral generator of a q-exponentially equicontinuous, analytic (g_α, g_1)-regularized resolvent family of angle $\delta'_\alpha \equiv \min((\frac{2}{\alpha}-1)\frac{\pi}{2}, \pi)$.

We close this section with the following observation. Keeping in mind the proof of Arendt-Widder theorem in SCLCSs [27, Theorem 2.1, p. 8] (cf. also [1]), we obtain the representation formulae for (a,k)-regularized C-resolvent families whose existence have been proved in the subordination principle [12, Theorem 2.11]. Here we would like to observe that it is not clear whether the above-mentioned result can be transferred to the class of q-exponentially equicontinuous (a,k)-regularized C-resolvent families in SCLCSs by means of these formulae and the method described in the proof of Theorem 2.1. Nevertheless, Theorem 3.1 enables one to prove a generalization of the subordination principle for a subclass of q-exponentially equicontinuous (a,k)-regularized resolvent families in complete locally convex spaces.

3. A generation result for q-exponentially equicontinuous (a,k)-regularized resolvent families and its consequences

The proofs of structural results given in [2] do not work any longer in the case of a general q-exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$. We must restrict ourselves to the case in which $C=I$ and (4) holds with $q_p=p$ (cf. also [2, Theorem 2.8]). In other words, we will consider a q-exponentially equicontinuous (a,k)-regularized resolvent family $(R(t))_{t\geq 0}$ which satisfies that, for every $p \in \circledast$, there exist $M_p \geq 1$ and $\omega_p \geq 0$ such that:

$$p(R(t)x) \leq M_p e^{\omega_p t} p(x),\ t \geq 0,\ x \in E. \tag{14}$$

In the sequel, the operator $\overline{R(t)_p}$ will be also denoted by $\overline{R_p}(t)$ $(t \geq 0)$.

We call a closed linear operator A acting on E compartmentalized (w.r.t. $\circledast$) if, for every $p \in \circledast$, $A_p := \{(\Psi_p(x), \Psi_p(Ax)) : x \in D(A)\}$ is a function ([2]). For example, every operator $T \in L_\circledast(E)$ is compartmentalized.

Theorem 3.1.

(i) *Suppose $a(t)$ satisfies (P1), $k(0) \neq 0$ and A is a subgenerator of a q-exponentially equicontinuous (a,k)-regularized resolvent family $(R(t))_{t\geq 0}$ which satisfies that, for every $p \in \circledast$, there exist $M_p \geq 1$ and $\omega_p \geq 0$ such that (14) holds. Then A is a compartmentalized operator and, for every $p \in \circledast$, $\overline{A_p}$ is a subgenerator of the exponentially bounded (a,k)-regularized resolvent family $(\overline{R_p}(t))_{t\geq 0}$ in $\overline{E_p}$ satisfying that:*

$$\left\|\overline{R_p}(t)\right\| \leq M_p e^{\omega_p t},\ t \geq 0. \tag{15}$$

Assume additionally that (2) holds. Then, for every $p \in \circledast$,

$$\overline{A_p}\int_0^t a(t-s)\overline{R_p}(s)\overline{x_p}ds = \overline{R_p}(t)\overline{x_p} - k(t)\overline{x_p},\ t \geq 0,\ \overline{x_p} \in \overline{E_p}, \tag{16}$$

the integral generator of $(R(t))_{t\geq 0}$ $((\overline{R_p}(t))_{t\geq 0})$, provided that $a(t)$ is kernel or that $k(t)$ satisfies (P1), is A $(\overline{A_p})$, and $(\overline{R_p}(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a,k)-regularized resolvent family of angle $\beta \in (0,\pi]$, provided that $(R(t))_{t\geq 0}$ is.

(ii) *Suppose $a(t)$ and $k(t)$ satisfy (P1), E is complete, A is a compartmentalized operator in E and, for every $p \in \circledast$, $\overline{A_p}$ is a subgenerator of an exponentially bounded (a,k)-regularized resolvent family $(\overline{R_p}(t))_{t\geq 0}$ in $\overline{E_p}$ satisfying (15)-(16). Then, for every $p \in \circledast$, (14) holds and A is a subgenerator of a q-exponentially equicontinuous (a,k)-regularized resolvent family $(R(t))_{t\geq 0}$ satisfying (2). Furthermore, $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a,k)-regularized resolvent family of angle $\beta \in (0,\pi]$ provided that, for every $p \in \circledast$, $(\overline{R_p}(t))_{t\geq 0}$ is a q-exponentially bounded, analytic (a,k)-regularized resolvent family of angle β.*

Proof. Suppose $x,\ y \in D(A)$ and $p(x) = p(y)$ for some $p \in \circledast$. Then $p(R(t)(x-y) + \int_0^t a(t-s)R(s)A(y-x)ds) = 0,\ t \geq 0$, which implies

$p(\int_0^t a(t-s)R(s)A(y-x)ds)=0$, $t\geq 0$. Therefore,

$$\int\limits_0^\infty e^{-\lambda t}\Psi_p\Big(\int\limits_0^t a(t-s)R(s)A(y-x)ds\Big)dt$$

$$=\int\limits_0^\infty e^{-\lambda t}\int\limits_0^t a(t-s)\Psi_p(R(s)A(y-x))dsdt=0,\ \Re\lambda>\max(abs(a),\omega_p),$$

and by the uniqueness theorem for the Laplace transform, $\Psi_p(R(t)A(x-y))=0$, $t\geq 0$. Using the fact that $(R(t))_{t\geq 0}$ is non-degenerate, we obtain that $p(A(x-y))=0$ and $p(Ax)=p(Ay)$, so that A_p is a linear operator in E_p. Let (x_n) be a sequence in $D(A)$ with $\lim_{n\to\infty}\Psi_p(x_n)=0$ and $\lim_{n\to\infty}\Psi_p(Ax_n)=y$ in $\overline{E_p}$. Then we have $\lim_{n\to\infty}p(\int_0^t a(t-s)R(s)Ax_nds)=\lim_{n\to\infty}||\int_0^t a(t-s)\Psi_p(R(s)Ax_n)ds||_{\overline{E_p}}=\lim_{n\to\infty}||\int_0^t a(t-s)\overline{R_p}(s)A_px_nds||_{\overline{E_p}}=0$, $t\geq 0$, which implies $0=\lim_{n\to\infty}\int_0^t a(t-s)\overline{R_p}(s)A_px_nds=\int_0^t a(t-s)\overline{R_p}(s)yds=0$, $t\geq 0$. Taking the Laplace transform, one obtains $\overline{R_p}(t)y=0$, $t\geq 0$ and, in particular, $y=0$ since $\overline{R_p}(0)=k(0)I$ and $k(0)\neq 0$. The above implies that A_p is a closable linear operator in $\overline{E_p}$ and that A is a compartmentalized operator in E. It is checked at once that $\overline{R_p}(t)\overline{A_p}\subseteq\overline{A_pR_p}(t)$, $t\geq 0$. Furthermore, (15) holds and the mapping $t\mapsto\overline{R_p}(t)x_p$, $t\geq 0$ is continuous for any $x_p\in E_p$, which implies by the standard limit procedure that the mapping $t\mapsto\overline{R_p}(t)\overline{x_p}$, $t\geq 0$ is continuous for any $\overline{x_p}\in\overline{E_p}$. The functional equality of $(R(t))_{t\geq 0}$ implies $\overline{R_p}(t)x_p-k(t)x_p=\int_0^t a(t-s)\overline{R_p}(s)A_px_pds$, $t\geq 0$, $x_p\in D(A_p)$, and therefore, $\overline{R_p}(t)\overline{x_p}-k(t)\overline{x_p}=\int_0^t a(t-s)\overline{R_p}(s)\overline{A_p}\overline{x_p}ds$, $t\geq 0$, $\overline{x_p}\in D(\overline{A_p})$. Hence, $\overline{A_p}$ is a subgenerator of the exponentially bounded, non-degenerate (a,k)-regularized resolvent family $(\overline{R_p}(t))_{t\geq 0}$ in $\overline{E_p}$. If (2) holds, then $\overline{R_p}(t)x_p-k(t)x_p=A_p\int_0^t a(t-s)\overline{R_p}(s)x_pds$, $t\geq 0$, $x_p\in E_p$, which implies (16). It is not difficult to see that the integral generator of $(R(t))_{t\geq 0}$ ($(\overline{R_p}(t))_{t\geq 0}$), provided that $a(t)$ is kernel or that $k(t)$ satisfies (P1), is A ($\overline{A_p}$). Suppose now that $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a,k)-regularized resolvent family of angle β. Then the mapping $z\mapsto\overline{R_p}(z)x_p$, $z\in\Sigma_\beta$ is analytic for any $p\in\circledast$ and $x_p\in E_p$, because the mapping $z\mapsto R(z)x$, $z\in\Sigma_\beta$ ($x\in E$) is infinitely differentiable and $\Psi_p(\cdot)$ is continuous. It is clear that the condition

$$p(R(z)x)\leq M_{p,\varepsilon}e^{\omega_{p,\varepsilon}|z|}p(x),\ x\in E,\ z\in\Sigma_{\beta-\varepsilon},\ p\in\circledast \tag{17}$$

for some $M_{p,\varepsilon} \geq 1,\ \omega_{p,\varepsilon} \geq 0$ and $\varepsilon \in (0, \beta)$ implies the following one:

$$\left\|\overline{R_p}(z)\right\| \leq M_{p,\varepsilon} e^{\omega_{p,\varepsilon}|z|},\ z \in \Sigma_{\beta-\varepsilon}. \tag{18}$$

Now the analyticity of the mapping $z \mapsto \overline{R_p}(z)\overline{x_p}$, $z \in \Sigma_\beta$ ($p \in ⊛$, $\overline{x_p} \in \overline{E_p}$) follows from Vitali's theorem [1, Theorem A.5]. Let $\delta \in (0, \beta)$. Then the mapping $z \mapsto \overline{R_p}(z)x_p$, $z \in \overline{\Sigma_\delta}$ ($p \in ⊛$, $x_p \in E_p$) is continuous, which implies by (18) the continuity of mapping $z \mapsto \overline{R_p}(z)\overline{x_p}$, $z \in \overline{\Sigma_\delta}$ ($p \in ⊛$, $\overline{x_p} \in \overline{E_p}$). The above implies that $(\overline{R_p}(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a, k)-regularized resolvent family of angle β ($p \in ⊛$). In order to prove (ii), notice first that the projective limit of $\{\overline{A_p} : p \in ⊛\}$ is A and that $(x, y) \in D(A)$ iff $(\Psi_p(x), \Psi_p(y)) \in \overline{A_p}$ for all $p \in ⊛$. Set, for every $p \in ⊛$, $\omega_p' := \max(abs(a), abs(k), \omega_p)$. By [11, Theorem 2.6], for every $p \in ⊛$, the following holds:

$$\tilde{k}(\lambda)(I - \tilde{a}(\lambda)\overline{A_p})^{-1}\overline{x_p} = \int_0^\infty e^{-\lambda t}\overline{R_p}(t)\overline{x_p}dt,\ \overline{x_p} \in \overline{E_p},\ \Re\lambda > \omega_p',\ \tilde{k}(\lambda) \neq 0.$$

Define $F_p : \{\lambda \in \mathbb{C} : \Re\lambda > \omega_p'\} \to L(\overline{E_p})$ by $F_p(\lambda)\overline{x_p} := \int_0^\infty e^{-\lambda t}\overline{R_p}(t)\overline{x_p}dt$, $\lambda \in D(F_p)$, $\overline{x_p} \in \overline{E_p}$ ($p \in ⊛$). Then $F_p(\cdot)$ is analytic and $F_p(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\overline{A_p})^{-1}$, provided $\Re\lambda > \omega_p'$ and $\tilde{k}(\lambda) \neq 0$. Suppose now $p,\ q \in ⊛$ and $p \gg q$. Then it is clear that $\pi_{qp}(\overline{R_p}(0)\overline{x_p}) = \overline{R_q}(0)\pi_{qp}(\overline{x_p})$, $\overline{x_p} \in \overline{E_p}$. Fix for a moment $t > 0$. Then, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \max(\omega_p', \omega_q')$ and $\tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0$, we have by [2, Lemma 4.1]:

$$\begin{aligned}
&\pi_{qp}\big(\tilde{k}(\lambda)(I - \tilde{a}(\lambda)\overline{A_p})^{-1}\overline{x_p}\big) \\
&= \pi_{qp}\Big(\frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\big(\frac{1}{\tilde{a}(\lambda)} - \overline{A_p}\big)^{-1}\overline{x_p}\Big) \\
&= \frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\Big(\frac{1}{\tilde{a}(\lambda)} - \overline{A_q}\Big)^{-1}\pi_{qp}(\overline{x_p}) \\
&= \tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\overline{A_q}\big)^{-1}\pi_{qp}(\overline{x_p}),\ \overline{x_p} \in \overline{E_p}.
\end{aligned}$$

The above implies $\pi_{qp}(F_p(\lambda)\overline{x_p}) = F_q(\lambda)\pi_{qp}(\overline{x_p})$, $\Re\lambda > \max(\omega_p', \omega_q')$, $\overline{x_p} \in \overline{E_p}$, and:

$$\pi_{qp}\Big(\frac{d^n}{d\lambda^n}F_p(\lambda)\overline{x_p}\Big) = \frac{d^n}{d\lambda^n}F_q(\lambda)\pi_{qp}(\overline{x_p}),\ \Re\lambda > \max(\omega_p', \omega_q'),\ \overline{x_p} \in \overline{E_p},\ n \in \mathbb{N}. \tag{19}$$

By the Post-Widder inversion formula ([1]) and (19), we get that:

$$\begin{aligned}\pi_{qp}\big(\overline{R_p}(t)\overline{x_p}\big) &= \lim_{n\to\infty}\pi_{qp}\Big((-1)^n n!^{-1}\big(\frac{n}{t}\big)^{n+1}\Big[\frac{d^n}{d\lambda^n}F_p(\lambda)\Big]_{\lambda=n/t}\overline{x_p}\Big)\\ &= \lim_{n\to\infty}(-1)^n n!^{-1}\big(\frac{n}{t}\big)^{n+1}\Big[\frac{d^n}{d\lambda^n}F_q(\lambda)\Big]_{\lambda=n/t}\pi_{qp}(\overline{x_p})\\ &= \overline{R_q}(t)\pi_{qp}(\overline{x_p}),\ \overline{x_p}\in\overline{E_p}.\end{aligned}$$

Hence, $\{\overline{R_p}(t) : p \in \circledast\}$ is a projective family of operators. Denote by $(R(t))_{t\geq 0} \subseteq L(E)$ the projective limit of the above family. Then it can be verified without any substantial difficulties that $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous (a,k)-regularized resolvent family which satisfies the required properties. Suppose now that, for every $p \in \circledast$, $(\overline{R_p}(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a,k)-regularized resolvent family of angle β and that, for every $\varepsilon \in (0,\beta)$, (18) holds. Using the equality $\pi_{qp}\big(\overline{R_p}(t)\overline{x_p}\big) = \overline{R_q}(t)\pi_{qp}(\overline{x_p}),\ t>0,\ \overline{x_p}\in\overline{E_p}$ and the fact that $\pi_{qp}(\cdot)$ is a continuous homomorphism from $\overline{E_p}$ onto $\overline{E_q}$, we obtain from the uniqueness theorem for analytic functions that $\pi_{qp}\big(\overline{R_p}(z)\overline{x_p}\big) = \overline{R_q}(z)\pi_{qp}(\overline{x_p}),\ z\in\Sigma_\beta,\ \overline{x_p}\in\overline{E_p}$. Therefore, $\{\overline{R_p}(z) : p \in \circledast\}$ is a projective family of operators $(z \in \Sigma_\beta)$. Define $R(z)$ as the projective limit of $\{\overline{R_p}(z) : p \in \circledast\}$ $(z \in \Sigma_\beta)$. Then the mapping $z \mapsto R(z)x$, $z \in \Sigma_\beta \cup \{0\}$ $(x \in E)$ is continuous on any closed subsector of $\Sigma_\beta \cup \{0\}$ and, for every $\varepsilon \in (0,\beta)$, there exist $M_{p,\varepsilon} \geq 1$ and $\omega_{p,\varepsilon} \geq 0$ such that (17) holds. Let $x \in E$ and let C be an arbitrary closed contour in Σ_β. Then, for every $p \in \circledast$, $\Psi_p\big(\oint_C R(z)x\,dz\big) = \oint_C \Psi_p(R(z)x)dz = \oint_C \overline{R_p}(z)\Psi_p(x)dz = 0$, which implies $\oint_C R(z)x\,dz = 0$. Hence, for every $x^* \in E^*$, $\oint_C \langle x^*, R(z)x\rangle dz = 0$ and the mapping $z \mapsto \langle x^*, R(z)x\rangle$, $z \in \Sigma_\beta$ is analytic by Morera's theorem. It follows that the mapping $z \mapsto R(z)x$, $z \in \Sigma_\beta$ is analytic, and the proof of theorem is completed through a routine argument. □

Remark 3.1. In order for the proof of Theorem 3.1(ii) to work, one has to identify the operator A with the projective limit of family $\{\overline{A_p} : p \in \circledast\}$. This can be done only in the case that the space E is complete.

Keeping in mind Theorem 3.1 and [12, Theorem 2.8, Theorem 3.6-Theorem 3.7], one can simply formulate the Hille-Yosida type theorems for (analytic) q-exponentially equicontinuous (a,k)-regularized resolvent families in complete locally convex spaces, provided that $a(t)$ and $k(t)$ satisfy (P1), and that $k(0) \neq 0$.

The proof of following result follows immediately from Theorem 3.1 and [15, Theorem 2.11-Theorem 2.12, Corollary 2.15, Remark 2.16].

Theorem 3.2. *Let E be complete.*

(i) *Suppose $z \in \mathbb{C}$, $B \in L_{\circledast}(E)$, A is densely defined and generates a q-exponentially equicontinuous (a,k)-regularized resolvent family $(R(t))_{t\geq 0}$ satisfying (14). Let (P1) hold for $a(t)$, $k(t)$, $b(t)$, let $\tilde{a}(\lambda)/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z$, $\Re\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$, for some $\omega > \max(abs(a), abs(k), abs(b))$ and let $k(0) \neq 0$. Suppose that, for every $p \in \circledast$, there exists a sufficiently large number $\mu_p > 0$ and a number $\gamma_p \in [0,1)$ such that:*

$$\int_0^\infty e^{-\mu_p t} p\Big(B \int_0^t b(t-s)R(s)x\,ds + zBR(t)x\Big)\,dt \leq \gamma_p p(x),\ x \in D(A).$$

Then the operator $A+B$ is the generator of a q-exponentially equicontinuous (a,k)-regularized resolvent family $(R_B(t))_{t\geq 0}$. Furthermore, for every $t \geq 0$ and $x \in D(A)$:

$$R_B(t)x = R(t)x + \int_0^t R_B(t-r)\Big(B \int_0^r b(r-s)R(s)x\,ds + zBR(r)x\Big)\,dr.$$

(ii) *Suppose $B \in L_{\circledast}(E)$, $l \in \mathbb{N}$, A is densely defined and generates a q-exponentially equicontinuous (a,k)-regularized resolvent family $(R(t))_{t\geq 0}$ satisfying (14). Let $k(0) \neq 0$, let $a(t)$ and $k(t)$ satisfy (P1) and let the following conditions hold:*

(ii.1) $A^j B \in L_{\circledast}(E)$, $1 \leq j \leq l$.

(ii.2) *There exist a function $b(t)$ satisfying (P1) and $z,\ \omega \in \mathbb{C}$ such that:*

$$\tilde{a}(\lambda)^{l+1}/\tilde{k}(\lambda) = \tilde{b}(\lambda) + z,\ \Re\lambda > \max(\omega, abs(a), abs(k)),\ \tilde{k}(\lambda) \neq 0.$$

(ii.3) $\lim_{\lambda\to+\infty} \int_0^\infty e^{-\lambda t}|a(t)|dt = 0$ *and* $\lim_{\lambda\to+\infty} \int_0^\infty e^{-\lambda t}|b(t)|dt = 0$.

Then $A+B$ is the generator of a q-exponentially equicontinuous (a,k)-regularized resolvent family $(R_B(t))_{t\geq 0}$.

(iii) *Suppose* $\alpha > 0$, A *is densely defined and generates a q-exponentially equicontinuous* (g_α, g_1)*-regularized resolvent family* $(R(t))_{t\geq 0}$ *satisfying (14). Assume exactly one of the following conditions:*

(iii.1) $\alpha \geq 1$ *and* $B \in L_{\circledast}(E)$.

(iii.2) $\alpha < 1$ *and* $A^j B \in L_{\circledast}(E)$, $0 \leq j \leq \lceil \frac{1-\alpha}{\alpha} \rceil$.

Then the operator $A+B$ *is the generator of a q-exponentially equicontinuous* (g_α, g_1)*-regularized resolvent family* $(R_B(t))_{t\geq 0}$. *Furthermore, if* $(R(t))_{t\geq 0}$ *is a q-exponentially equicontinuous, analytic* (g_α, g_1)*-regularized resolvent family of angle* $\beta \in (0, \pi/2]$, *then* $(R_B(t))_{t\geq 0}$ *is.*

Concerning Theorem 3.2(iii), it is worthwhile to mention that the assertion of [15, Corollary 2.15] (cf. also [13, Theorem 2.5.7-Theorem 2.5.8]) does not admit a satisfactory reformulation for q-exponentially equicontinuous $(g_\alpha, g_{\alpha\beta+1})$-regularized C-resolvent families in Fréchet spaces, unless $C = I$ and $\beta = 0$.

Example 3.1.

(i) Let $\alpha \in (0,1)$. Set $a_\alpha(t) := \mathcal{L}^{-1}(\frac{\lambda^\alpha}{\lambda+1})(t)$, $t \geq 0$, $k_\alpha(t) := e^{-t}$, $t \geq 0$ and $\delta_\alpha := \min(\frac{\pi}{2}, \frac{\pi\alpha}{2(1-\alpha)})$. Suppose E is complete, $f \in L^1_{loc}([0,\infty) : E)$ and A is the integral generator of a q-exponentially equicontinuous $(C_0, 1)$-semigroup $(R(t))_{t\geq 0}$ satisfying (14). Then Theorem 3.1 combined with the analysis given in [15, Example 3.7] implies that A is the integral generator of a q-exponentially equicontinuous, analytic (a_α, k_α)-regularized resolvent family of angle δ_α, which can be applied (cf. [20]-[21] and [15] for more details) in the study of qualitative properties of the abstract Basset-Boussinesq-Oseen equation:

$$u'(t) - A\mathbf{D}_t^\alpha u(t) + u(t) = f(t),\ t \geq 0,\ u(0) = 0,$$

describing the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity.

(ii) Put $E := \{f \in C^\infty([0,\infty)) : \lim_{x\to+\infty} f^{(k)}(x) = 0 \text{ for all } k \in \mathbb{N}_0\}$ and $||f||_k := \sum_{j=0}^k \sup_{x\geq 0} |f^{(j)}(x)|$, $f \in E$, $k \in \mathbb{N}_0$. Then the topology induced by these norms turns E into a Fréchet space. Suppose $c_0 > 0$, $\beta > 0$, $s > 1$, $l > 0$ and define the operator A by $D(A) :=$

$\{u \in E : c_0u'(0) = \beta u(0)\}$ and $Au := c_0u''$, $u \in D(A)$. Then A cannot be the generator of a C_0-semigroup since $D(A)$ is not dense in E ([10]). Put $A_1 := A/c_0$, $\omega_{l,s}(\lambda) := \prod_{p=1}^{\infty}(1+\frac{l\lambda}{p^s})$, $\lambda \in \mathbb{C}$ and $k_{l,s}(t) := \mathcal{L}^{-1}(\frac{1}{\omega_{l,s}(\lambda)})(t),\ t \geq 0$. Using the well-known estimates for associated functions ([13]) and [11, (2.36)], we infer that there exists a constant $c_1 > 0$ such that, for every $\epsilon \in (0, \pi)$,

$$|\omega_{l,s}(\lambda)| \geq \exp\left(c_1\left(l(1+\cot\epsilon)^{-1}|\lambda|\right)^{1/s}\right),\ \lambda \in \Sigma_{\pi-\epsilon}. \tag{20}$$

Furthermore, $0 \in \operatorname{supp} k_{l,s}$, $k_{l,s}(0) = 0$ and $k_{l,s}(t)$ is infinitely differentiable in $t \geq 0$. We will prove that A is the integral generator of an equicontinuous analytic $k_{l,s}$-convoluted semigroup of angle $\pi/2$ and that there does not exist $n \in \mathbb{N}$ such that A is the integral generator of an exponentially equicontinuous n-times integrated semigroup on E (cf. also the proofs of [5, Theorem 4.1-Theorem 4.2, pp. 384–386]). It is checked at once that the operator $\lambda - A$ is injective for all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Let $\lambda = re^{i\theta}$ $(r > 0,\ |\theta| < \pi)$, $f \in E$ and $\mu = \lambda^{1/2}$. Then de L'Hospital's rule implies that, for every $k \in \mathbb{N}_0$, the C^∞-functions $x \mapsto u_{1,k}(x) := \int_0^x e^{-\mu(x-s)}f^{(k)}(s)ds = e^{-\mu x}\int_0^x e^{\mu s}f^{(k)}(s)ds$, $x \geq 0$ and $x \mapsto u_{2,k}(x) := \int_x^\infty e^{\mu(x-s)}f^{(k)}(s)ds = e^{\mu x}\int_x^\infty e^{-\mu s}f^{(k)}(s)ds$, $x \geq 0$ tend to 0 as $x \to +\infty$. Taken together with the computation given in the proof of the estimate (23), the above implies that the function

$$u(x) := \frac{1}{2\mu}\Big[\int_0^x e^{-\mu(x-s)}f(s)ds + \int_x^\infty e^{\mu(x-s)}f(s)ds\Big],\ x \geq 0,$$

belongs to E. Now it readily follows that the function $\omega(x) := u(x) + [\frac{c_0\mu-\beta}{c_0\mu+\beta}\frac{1}{2\mu}\int_0^\infty e^{-\mu s}f(s)ds]e^{-\mu x} := u(x) + \kappa(\mu, f)e^{-\mu x}$, $x \geq 0$, belongs to $D(A_1)$ and that $(\lambda - A_1)\omega = f$; therefore, $\lambda \in \rho(A_1)$ and $(\lambda - A_1)^{-1}f = \omega$. Direct computation shows that

$$\sup_{x\geq 0}|u(x)| \leq \frac{\sup_{x\geq 0}|f(x)|}{|\lambda|\cos\frac{\theta}{2}} \text{ and } \sup_{x\geq 0}|u_{1,k}(x)| \leq \frac{\sup_{x\geq 0}|f^{(k)}(x)|}{|\lambda|^{1/2}\cos\frac{\theta}{2}},\ k \in \mathbb{N}_0. \tag{21}$$

Now we obtain that there exists an absolute constant $c > 0$ such that, for every $n \in \mathbb{N}$,

$$\big\|(\lambda-A_1)^{-1}f\big\|_n=\sum_{j=0}^{n}\sup_{x\geq 0}\big|u^{(j)}(x)+\kappa(\mu,f)(-1)^j\mu^j e^{-\mu x}\big|$$
$$\leq \sum_{j=1}^{n}\Big\{\sup_{x\geq 0}\Big|\frac{1}{2\mu}\Big[\int_0^x e^{-\mu(x-s)}f^{(j)}(s)\,ds+\sum_{l=0}^{j-1}(-1)^l\mu^l f^{(j-1-l)}(0)e^{-\mu x}$$
$$-\sum_{l=1}^{j}\sum_{l_0=0}^{l-1}\binom{j}{l}\mu^{j-l}\binom{l-1}{l_0}(-1)^{l-1-l_0}\mu^{l-1-l_0}f^{(l_0)}(x)$$
$$+\mu^j\int_x^{\infty}e^{\mu(x-s)}f(s)\,ds\Big]+\kappa(\mu,f)(-1)^j\mu^j e^{-\mu x}\Big|\Big\}+\frac{c||f||_0}{|\lambda|\cos\frac{\theta}{2}}$$
$$\leq \frac{c||f||_0}{|\lambda|\cos\frac{\theta}{2}}+\sum_{j=1}^{n}\Big[\frac{||f||_j}{2|\lambda|\cos\frac{\theta}{2}}+j\big(|\mu|^{-1}+|\mu|^{j-1}\big)||f||_{j-1}\Big]$$
$$+\sum_{j=1}^{n}\Big[4^{j-1}\big(|\lambda|^{(-1)/2}+|\lambda|^{(j-1)/2}\big)||f||_{j-1}$$
$$+\frac{1}{2}|\lambda|^{(j-2)/2}\frac{||f||_0}{\cos\frac{\theta}{2}}+\frac{c|\mu|^j||f||_0}{|\lambda|\cos\frac{\theta}{2}}\Big]$$

$$\leq n||f||_n\frac{1}{2|\lambda|\cos\frac{\theta}{2}}+2n^2\big(|\mu|^{-1}+|\mu|^{n-1}\big)||f||_{n-1}+\frac{2cn||f||_0\big(1+|\mu|^n\big)}{|\lambda|\cos\frac{\theta}{2}}$$

(22)

$$+n4^{n-1}\big(|\lambda|^{1/2}+|\lambda|^{(n-1)/2}\big)||f||_{n-1}+n\big(1+|\lambda|^{n/2}\big)\frac{||f||_0}{2|\lambda|\cos\frac{\theta}{2}}.$$

The inequality $\exp(-\zeta x^{1/s})x^{\eta}\leq (s\eta/\zeta)^{\eta s}$, $x>0$, $\zeta>0$, $\eta>0$ combined with (20)-(22) implies that, for every $\epsilon\in(0,\pi)$, the family $\{\lambda\widetilde{k_{l,s}}(\lambda)(\lambda-A)^{-1}:\lambda\in\Sigma_{\pi-\epsilon}\}$ is equicontinuous. Moreover, $\lim_{\lambda\to+\infty}\lambda\widetilde{k_{l,s}}(\lambda)(\lambda-A)^{-1}f=0=k_{l,s}(0)f$, $f\in E$. By [12, Theorem 3.7] and its proof, it follows that A is the integral generator of an equicontinuous analytic $k_{l,s}$-convoluted semigroup $(R(t))_{t\geq 0}$ of angle $\pi/2$ satisfying additionally that, for every $k\in\mathbb{N}_0$ and $\epsilon\in(0,\pi)$, there exists $c(k,\epsilon)>0$ such that $||R(z)f||_k\leq c(k,\epsilon)||f||_k$, $z\in\Sigma_{\pi-\epsilon}$, $f\in E$. Assume that there exists $n \in \mathbb{N}$ such that A is the integral generator of an exponentially equicontinuous n-times integrated semigroup on E. Without loss of generality, we may assume that $2n+3 > 2\beta/c_0$. Then there exists a sufficiently large $\nu > 0$ such that the family $\{\lambda^{-n}(\lambda - A)^{-1} : \lambda > \nu\}$ is equicontinuous, which simply implies that there exist $c_n > 0$ and $n' \in \mathbb{N}$ with:

$$\sup_{x\geq 0}\Big|\frac{1}{2\lambda^{1/2}}\Big[\int_0^x e^{-\lambda^{1/2}(x-s)}f^{(2n+5)}(s)ds$$

$$+\sum_{j=0}^{2n+4}(-1)^j\lambda^{j/2}f^{(2n+4-j)}(0)e^{-\lambda^{1/2}x}\Big]$$

$$+\frac{1}{2}\Big[\lambda^{n+2}\int_x^\infty e^{\lambda^{1/2}(x-s)}f(s)ds-\sum_{l=1}^{2n+5}\sum_{l_0=0}^{l-1}\binom{2n+5}{l}\lambda^{(2n+3-l_0)/2}$$

$$\times(-1)^{l-1-l_0}\binom{l-1}{l_0}f^{(l_0)}(x)\Big]-\Big[\frac{c_0\lambda^{1/2}-\beta}{c_0\lambda^{1/2}+\beta}\frac{1}{2\lambda^{1/2}}\int_0^\infty e^{-\lambda^{1/2}s}f(s)ds\Big]$$

(23)

$$\times\lambda^{(2n+5)/2}e^{-\lambda^{1/2}x}\Big|\leq c_n\lambda^n\big\|f\big\|_{n'},\ \lambda>\nu,\ f\in E.$$

Denote by $g_f(x,\lambda)$ the function whose supremum appears in (23). Since $\sum_{l=1}^{2n+5}\binom{2n+5}{l}(-1)^{l-1}=1$ and $\sum_{l=1}^{2n+5}\binom{2n+5}{l}(l-1)(-1)^l=1$, it can be easily seen that there exists a sufficiently large number $a_n > 0$, depending only on n, such that:

$$2\sup_{x\geq 0}g_{e^{-\cdot}}(x,\lambda)\geq 2g_{e^{-\cdot}}(0,\lambda)$$

$$\geq\lambda^n\Big|2\lambda+\lambda^{1/2}+\sum_{l=1}^{2n+5}\binom{2n+5}{l}\binom{l-1}{2}(-1)^l\lambda^{1/2}$$

$$+\frac{2\beta\lambda^2}{(c_0\lambda^{1/2}+\beta)(1+\lambda^{1/2})}\Big|-a_n\lambda^n,\ \lambda>a_n,$$

which implies $\lim_{\lambda\to+\infty}\lambda^{-n}\sup_{x\geq 0}g_{e^{-\cdot}}(x,\lambda)=+\infty$. A contradiction. The question whether there exists an injective operator $C \in L(E)$ such that A is the integral generator of an exponentially equicontinuous C-regularized semigroup on E is non-trivial and will not be further discussed in the context of this paper; let us only mention in passing that the existence of such an operator C can be proved only with the help of results concerning ultradistribution semigroups in locally convex spaces (cf. [13, Subsection 3.6.2] for the Banach space case, and [25]). Now we will explain how one can use the obtained result in the analysis of a control problem for a one-dimensional heat equation for materials with memory (cf. [23, pp. 146-147]). Let $L^1_{loc}([0,\infty)) \ni a$ satisfy (P1), let $abs(a)=0$ and let the analytic function $\hat{a} : \mathbb{C}\backslash(-\infty,0] \to \mathbb{C}\backslash(-\infty,0]$ satisfy $\hat{a}(\lambda)=\tilde{a}(\lambda)$, $\Re\lambda>0$. Suppose that $\sigma\in(0,1)$ and, for every $\epsilon\in(0,\pi)$ and $n\in\mathbb{N}$, there exist $c^1_{\epsilon,n}>0$ and $c^2_{\epsilon,n}>0$ such that $|\hat{a}(\lambda)|^{-n}\leq c^1_{\epsilon,n}\exp(c^2_{\epsilon,n}|\lambda|^{\sigma})$, $\lambda\in\Sigma_{\pi-\epsilon}$. By [12, Theorem 3.7], we get that, for every $l>0$ and $s\in(1,1/\sigma)$, the operator A is the integral generator of an equicontinuous analytic $(a,k_{l,s})$-regularized resolvent family $(S(t))_{t\geq0}$ of angle $\pi/2$ satisfying additionally that, for every $k\in\mathbb{N}_0$ and $\epsilon\in(0,\pi)$, there exists $c(k,\epsilon)'>0$ with $||S(z)f||_k\leq c(k,\epsilon)'||f||_k$, $z\in\Sigma_{\pi-\epsilon}$, $f\in E$. This, in particular, implies the existence of regularized solutions to the problem [23, (5.68), p. 147].

(iii) Suppose $E=L^2(\mathbb{R}^n)$, $0\leq l\leq n$ and $1\leq\alpha\leq2$. Put $\mathbb{N}^l_0:=\{\eta\in\mathbb{N}^n_0:\eta_{l+1}=\cdots=\eta_n=0\}$ and recall that the space E_l is defined by $E_l:=\{f\in E: f^{(\eta)}\in E \text{ for all } \eta\in\mathbb{N}^l_0\}$. The totality of seminorms $(q_\eta(f):=\sum_{\mu\leq\eta}||f^{(\mu)}||_{L^2(\mathbb{R}^n)},\ f\in E_l;\ \eta\in\mathbb{N}^l_0)$ induces a Fréchet topology on E_l ([27]). Let $a_\eta\in\mathbb{C}$, $0\leq|\eta|\leq N$, let $P(x)=\sum_{|\eta|\leq N}a_\eta x^\eta$, $x\in\mathbb{R}^n$, and let $\omega\geq0$ satisfy $\sup_{x\in\mathbb{R}^n}\Re(P(x)^{1/\alpha})\leq\omega$. Suppose that the operator $P(D)f:=\sum_{|\eta|\leq N}a_\eta(-i)^{|\eta|}f^{(\eta)}$ acts on E_l with its maximal distributional domain. Then we know that $P(D)$ generates an exponentially equicontinuous (g_α,g_1)-regularized resolvent family $(R_\alpha(t))_{t\geq0}$ in the space E_l (cf. [12, Example 3.17] and [14, Remark 2.2]) and that there exists a constant $M\geq1$ such that:

$$q_\eta\big(R_\alpha(t)f\big)\leq Me^{\omega t}q_\eta(f),\ t\geq0,\ f\in E_l,\ \eta\in\mathbb{N}^l_0.$$

Let $\varphi\in C^\infty(\mathbb{R}^n)$ possess bounded derivatives of all orders and let $(Bf)(x):=\varphi(x)f(x)$, $f\in E_l$, $x\in\mathbb{R}^n$. Then $B\in L_{\circledast}(E_l)$ and, by Theorem 3.2(iii), the operator $P(D)+B$ generates a q-exponentially equicontinuous (g_α,g_1)-regularized resolvent family $(R^B_\alpha(t))_{t\geq0}$ in the space E_l.

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